Algebraic Model for the Dualism of Selective and Structural Manifestations of Information (Logics, Algebras and Languages in Computer Science)

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1. Introduction

An authentic theory of information can be developed only when the concept of information is clearly and properly defined. Although there is a common agreement regarding the importance of this concept and its applicability in a very wide range of contexts is widely recognized, the term "information" is usually used without much effort to specify its meaning or relationship to the relevant conceptual framework. The dominating, fallacious view is that the generality of the concept of information calls for its definition in terms of simple, "obvious", common sense concepts.

Attempts to define information, although multiple, were usually not very successful, producing concepts too narrow to be applied in all domains where the term is used, too vague to meet the requirements of a proper definition, or too much detached from the philosophical or scientific tradition to provide foundations for a nontrivial reflection or theoretical analysis. One of the typical deficiencies of these attempts was the lack of connection between two major aspects of information, its quantitative and qualitative characteristics.

The attempts related to the quantitative methods initiated by Shannon in his study of communication ignored the structural (qualitative) aspects of information. No wonder that they had to give up the study of semantics for information, if in such perspective information is an amorphous aggregate with the description exclusively in terms of the probability of meaningless components. However, the attempts focusing on the qualitative aspects of information were even less successful, since they did not go much beyond the relationship with the philosophical reflection on the concept of form.

In order to combine both aspects of information and to place this concept in the context of non-trivial philosophical conceptual framework, the present author introduced his definition of information in terms of the one-many categorical opposition with a very long and rich philosophical tradition [1]. Thus, information is defined as a resolution of the one-many opposition, or in other words as that, which makes one out of many. There are two ways in which many can be made one, either by the selection of one out of many, or by binding the many into a whole by some structure. The former is a selective manifestation of information and the latter is a structural manifestation. They are different manifestations of the same concept of information, not different types, as one is always accompanied by the other, although the multiplicity (many) can be different in each case.

This dualism between coexisting manifestations was explained by the author in his earlier presentations of the definition using a simple example of the collection of the keys to rooms in a hotel. It is easy to agree that the use of keys is based on their informational content, but information is involved in this use in two different ways, through the selection of the right key, or through the geometric description of its shape. We can have numbers of the rooms attached to keys which allow a selection of the appropriate key out of many other placed on the shelf. However, we can also consider the shape of key's feather made of mechanically distinguishable elements or even of molecules. In the latter case, geometric structure of the key is carrying information. The two manifestations of information make one out of very different multiplicities, but they are closely interrelated.

The definition of information presented above, which generalizes many earlier attempts and which due to its very high level of abstraction can be applied to practically all instances of the use of the term information, can be used to develop a mathematical formalism for information. It is not a surprise, that the formalism is using very general framework of algebra [2].

The concept of information requires a variety (many), which can be understood as an arbitrary set S (called a carrier of information). Information system is this set S equipped with the family of subsets $\mathfrak{J}$ satisfying conditions: entire S is in $\mathfrak{J}$, and together with every subfamily of $\mathfrak{J}$, its intersection belongs to $\mathfrak{J}$, i.e. $\mathfrak{J}$ is a Moore family. Of course, this means that we have a closure operator defined on S (i.e. a function f on the power set 2^S of a set S such that: (1) For every subset A of S, $A \subseteq f(A)$; (2) For all subsets A, B of S, $A \subseteq B \Rightarrow f(A) \subseteq f(B)$; (3) For every subset A of S, $f(f(A)) = f(A)$). The Moore family $\mathfrak{J}$ of subsets is simply the family f-Cl of all closed subsets, i.e. subsets A of S such that $A= f(A)$. The family of closed subsets $\mathfrak{J}$ = f-Cl is equipped with the structure of a complete lattice L_f by the set theoretical inclusion. L_f can play a role of the generalization of logic for not necessarily linguistic information systems, although it does not have to be a Boolean algebra. In many cases it maintains all fundamental characteristics of a logical system [3].

Information itself is a distinction of a subset $\mathfrak{J}_0$ of $\mathfrak{J}$, such that it is closed with respect to (pair-wise) intersection and is dually-hereditary, i.e. with each subset belonging to $\mathfrak{J}_0$, all subsets of S including it belong to $\mathfrak{J}_0$ (i.e. $\mathfrak{J}_0$ is a filter in L_f).

The Moore family $\mathfrak{J}$ can represent a variety of structures of a particular type (e.g. geometric, topological, algebraic, logical, etc.) defined on the subsets of S. This corresponds to the structural manifestation of information. Filter $\mathfrak{J}_0$ in turn, in many mathematical theories associated with localization, can be used as a tool for identification, i.e. selection of an element within the family $\mathfrak{J}$, and under some conditions in the set S. For instance, in the context of Shannon's selective information based on a probability distribution of the choice of an element in S, $\mathfrak{J}_0$ consists of elements in S which have probability measure 1, while $\mathfrak{J}$ is simply the set of all subsets of S.

This approach combines both manifestations of information, but the relationship between articulations of these manifestations within the formalism thus far was based on the intuitive interpretation. It was not clear in what sense we can talk about dualism. What exactly do we mean by dualism? How can we use the formalism of information to describe two information systems in the dual relationship?

The last question is of special importance, as the concept of dualism was used by the author to consider dynamics of information and computing as well as other hierarchically organized information systems [4, 5].

This paper is proposing answers to such questions based on an algebraic model of dualism.

2. Preliminaries

This paper will refer to the two main algebraic concepts. The first, concept of a Gallois connection is well known and belongs to the earliest tools of closure space theory [6]. The second, concept of a frame for the closure space was introduced by the author in a relatively recent article in the context of information integration [7]. While Galois connections are discussed in many introductory texts about lattices and a brief overview is provided here just for reader's convenience and for the clarification of symbolic conventions, frames require more extensive explanation.

For our purpose, we can focus on the concept of polarity, i.e. Galois connection generated by a binary relation. Let S, T be sets, and $R \subseteq S \times T$ be a binary relation between sets S and T. R^* is the converse relation of R, i.e. the relation $R^* \subseteq T \times S$ such that $\forall x \in S\ \forall y \in T$: xRy iff yR^*x. Then we define $R^a(A) = \{y \in T: \forall x \in A: xRy\}$, $R^e(A) = \{y \in T: \exists y \in A: xRy\}$. We can simplify our notation for single element subsets: $R(x) = R^a(\{x\}) = R^e(\{x\})$. Obviously, $R^a(A) = \cap\{R(x): x \in A\}$ and $R^e(A) = \cup\{R(x): x \in A\}$.

If R is a binary relation between sets S and T, the pair of functions $\varphi: 2^S \to 2^T$ and $\psi: 2^T \to 2^S$ between the power sets of S and T defined on subsets A of S by $\varphi: A \to R^a(A)$ and on subsets B of T by $\psi: B \to R^{*a}(B)$ forms a Galois connection, and therefore their both compositions, defined on subsets A of S by $f(A) = \varphi\psi(A) = R^aR^{*a}(A)$ and on subsets B of T $g(B) = \varphi\psi(B) = R^{*a}R^a(B)$ are transitive closure operators. Also, the functions φ, ψ are dual isomorphisms between the lattices L_f and L_g of closed subsets for the closure operators f and g.

Probably the best known example of the use of closure operators generated by polarity is Dedekind's "completion by cuts". In this case we have a partially ordered set $<S, \leq>$ in which:

$\leq^a(A) = \{x \in S: \forall y \in A, y \leq x\}$, $\leq^{*a}(A) = \{x \in S: \forall y \in A, y \geq x\}$ and the closure operator f defined by $f_c(A) = \leq^{*a}\leq^a(A)$. This gives us MacNeille's completion [6]:

Let $[P, \leq]$ *be a poset and* φ *a function from P to the complete lattice* L_c *of the* f_c *-closed subsets of P defined by* $\varphi(x) = \leq^{*a}\leq^a(\{x\})$. *Then* φ *is an injective, isotone and inverse-isotone function preserving all suprema and infima that happen to exist in the poset* $[P, \leq]$.
MacNeille's completion is isomorphic to the original poset whenever it is already a complete lattice.

The concept of a frame for the closure space defined by a transitive closure operator f on a set S appeared in the context of the reconstruction of the entire closure space from its subset. To avoid misunderstanding, it has to be emphasized that frames are very different from generating sets within closure space $<S,f>$ defined by $B \in$ f-Gen if $f(B) = S$. Bases, i.e. generating and independent subsets ($B \in$ f-Ind if $\forall x \in B: x \notin f(B \setminus \{x\})$), which in vector spaces are minimal generating subsets are of great importance for linear algebra. However, in more general cases bases are of limited interest for the study of closure spaces. Bases may have different cardinality, or may not exist at all.

Example 2.1. Let S be an infinite set and f be a closure operator defined by $f(A) = A$ if A is a finite subset of S, and $f(A) = S$ otherwise. It is easy to see that there is no base in this closure space, as there is no minimal infinite subset.

Example 2.2. Let X be a set with two disjoint proper, nonempty subsets T and U, and let the closure operator f be defined by $f(A) = S$, if $T \subseteq A$ or $U \subseteq A$, $f(A) = A$ otherwise. Then, both T and U are bases. If T and U are not equicardinal, they provide example of bases of different cardinality in the same closure space.

Thus, our objective is to identify the subsets of a closure space $<S,f>$ from which we can reconstruct the entire structure, in similar way as a complete atomic Boolean algebra or quantum logic can be reconstructed from the set of its atoms, a finite distributive lattice from the set of its join-irreducible elements, or more generally, a lattice without infinite chains from the set consisting of all join- and meet-irreducible elements. Example 2.2 showed that the distinction of a base does not give us much information about the closure operator, as the subset T is a base, but the closure for many subsets is not determined by this fact.

For this reason, we will focus our attention on frames defined as follows.

Definition 2.1. *Let $<S,f>$ be a closure space and B its subset (not necessarily proper). B is a frame for $<S,f>$, if $\forall A \subseteq S\ \exists B_A \subseteq B: f(A) = f(B_A)$. A frame is proper, if B is a proper subset of S; it is a minimal frame, if there is no proper subset of B which is a frame. The closure space is simple if it does not have proper frame.*

Proposition 2.1. [7] *The condition defining a frame for closure space $<S,f>$ is equivalent to: $\forall A \subseteq S: f(A) = f(B \cap f(A))$, where the equality can be replaced by the inclusion $\subseteq$.*

Since all set S is always a (trivial) improper frame, each closure space has at least one frame. In some cases it is the only frame, as can be seen in the closure space from Example 2.1 above, in which the only subset of S having nonempty intersections (and therefore nonempty closure of the intersections) with all (closed) one-element sets is the set S.

Naturally, we are interested in the minimal subsets of closure spaces which are frames. It is obvious that $B \setminus f(\emptyset)$ is always a frame, whenever B is a frame in $<S,f>$.

Although, obviously in every finite closure space there exists a minimal frame, in infinite spaces there may be no minimal frames at all, as the following example shows.

Example 2.3. Let $S = [0,1]$ the interval of real numbers with its natural ordering $\leq$, and $f(A) = \leq^{*a}\leq^{a}(A)$. $<S,f>$ has a nontrivial frame B consisting of all rational numbers on [0,1]. The closed subsets for this closure are simply closed intervals [0,a] for every real number a in [0,1], and therefore the condition for being a frame is for the set B: $\forall a \in S: [0,a] = f(B \cap [0,a])$. If a is rational, it is obvious. If it is irrational, then $B \cap [0,a]$ consists of all rational numbers on the interval [0,a], whose closure is [0,a].

Now, this closure space does not have a minimal frame. This can be restated as follows. If B is a frame, then $\forall a \in S: B_a = B \setminus \{a\}$ is also a frame. To prove it, suppose there exists a subset A of S, such that $f(A) \neq f(B_a \cap f(A))$. This means $f(A) = [0,d]$ for some d in S, $f(B_a \cap f(A)) = [0,c]$ for some c in S, and $c < d$. Since $B_a \cap f(A) = B_a \cap [0,d] \subseteq f(B_a \cap f(A)) = [0,c]$, we have $B_a = B \setminus \{d\}$, as otherwise $f(A) = f(B_a \cap f(A))$, and therefore $a = d$.

Now, let $c < e < d = a$. Then, $B \cap [0,e] = B_a \cap [0,e]$, and $f(B \cap [0,e]) = f(B_a \cap [0,e]) \subseteq f(B_a \cap [0,d]) = [0,c]$, and therefore $f(B \cap [0,e]) \neq [0,e]$, which means B is not a frame, a contradiction concluding the proof.

We already have identified a convenient equivalent for the definition of a frame in Proposition 2.1. However, there are several other equivalent conditions for a subset B of the closure space $<S,f>$ to be a frame.

Proposition 2.2. [7] *The following conditions for a subset B of S are equivalent:*

a) B is a frame for $<S,f>$,

b) $\forall A \subseteq S: A \subseteq f(B \cap f(A))$,

c) $\forall x \in S: x \in f(B \cap f(\{x\}))$,
d) $\forall x \in S: f(\{x\}) = f(B \cap f(\{x\}))$,
e) $\forall C,D \subseteq S: f(C) \cap B = f(D) \cap B \Rightarrow f(C) = f(D)$,
f) $\forall C \subseteq S: C \in f\text{-}Cl \Rightarrow f(B \cap C) = C$.

Remark. As a consequence of the fourth condition, every T_1 closure space $<S,f>$ (in which every one-element subset, or singleton, is closed, and therefore the empty set is also closed) must be simple, as the only set intersecting with all singletons is all set S.

Also, it is worth to mention that in the fifth condition the arbitrary subsets C and D cannot be replaced by singletons, as the following example shows.

Example 2.4. [7] Let $<S,f>$ be a closure space, T be a proper, nonempty subset of S, and f be defined for any subset A of S by: $f(A) = A$ if $A \subseteq T$, and $f(A) = S$ otherwise. Then, The set T satisfies the condition $\forall x,y \in S$: $f(\{x\}) \cap T = f(\{y\}) \cap T \Rightarrow f(\{x\}) = f(\{y\})$, but T is not a frame for $<S,f>$.

The following two examples justify our earlier claim that the concept of a frame has a similar role in closure spaces to the role played by the set of atoms in a finite Boolean algebra or the set of join-irreducible elements in a finite distributive lattice.

Example 2.5. Let L be a finite set with a structure of Boolean algebra defined on it. We can define a closure space structure on L by $f(A) = \leq^{*a}\leq^{a}(A)$. Then, the set At(L) of all atoms in L is a minimal frame. Moreover, the restriction of the closure operator f to At(L) has all subsets of At(L) as closed subsets.

Example 2.6. Let L be a finite set with a structure of distributive lattice defined on it. As above, $f(A) = \leq^{*a}\leq^{a}(A)$. Then, the set J(L) of all join-irreducible elements (elements which are not joins of other elements) is a minimal frame.

Let's return to the issue of the relationship between frames and bases in closure spaces. Obviously, every frame B is a generating set ($f(B) = S$). However, it does not have to be an independent set, and therefore it does not have to be a base. Since independent sets are minimal sets generating their closure it follows that if a frame is a base, it has to be a minimal frame. However, Example 2.2 shows an example of a simple closure space (with only trivial frame of entire set S) in which there are proper subsets T and U which are bases. Thus the concepts of frames and of bases are essentially different, although not mutually exclusive. The following is a simple example of a frame which is a base.

Example 2.7. [7] Let $\{T_i: i \in I\}$ be a partition of S, and the closure operator f on S be defined by $f(A) = \cup\{T_i: i \in I \text{ and } T_i \cap A \neq \varnothing\}$. Then every subset B of S, such that $\forall i \in I: |B \cap T_i| = 1$ is a minimal frame which also is a base.

It is possible to characterize the subsets which are both frames and bases.

Proposition 2.3. [7] *A frame B in a closure space $<S,f>$ is a base iff the restriction g of the closure operation f to set B satisfies $\forall A \subseteq B: g(A) = A$.*

Since frames are closed subsets only in simple spaces, the restriction of the closure operator to a frame is usually not a subspace, i.e. the action of the restricted closure operator on the subsets of the frame is different from the action of the original closure operator. However, the restriction turns out to be in a strict correspondence with the original closure space structure.

In the following, if no confusion is likely, g (with possible indices) will indicate the restriction of the closure operator f (with possible corresponding indices) to the frame B in <S,f>.

We could see above that the action of a closure operator f on an entire space S can be recovered from its action on any frame B. However, it was necessary to know the closures of all subsets of B within the original space <S,f>. The question is to what extent the restriction of a closure operator f on set S to a frame B determines the original structure.

We will start from considering the relationship between the lattices of closed subsets for <S,f> and <B,g>.

Proposition 2.4. [7] *Let L_f and L_g be the complete lattices of closed subsets in <S,f> and <B,g> respectively, and g be the restriction of closure f to a frame B in <S,f>. Then L_f and L_g are isomorphic.*

Corollary. *Let f_1 and f_2 be closure operators defined on the same set S, g_1 and g_2 be their respective restrictions to a subset B of S, which is a frame for both closure spaces <S,f_1> and <S,f_2>. Then, from the equality of the restrictions $g_1 = g_2$ follows the isomorphism of the lattice of f_1-closed subsets of S and the lattice of f_2-closed subsets of S.*

However, from the equality of restrictions we can only conclude about the isomorphism of the lattices of closed subsets for the original closure spaces, but not about their identity. Thus, frames determine the structure of entire closure space only up to isomorphism of lattices of closed subsets, as the following example shows.

Example 2.8. Let f_1 and f_2 be closure operators defined as in Example 2.7 on the same set S by two partitions $\{T_i: i \in I\}$ and $\{U_i: i \in I\}$, which although different satisfy the condition $\forall i \in I$: $T_i \cap U_i \neq \varnothing$. Then we can select a set B which satisfies both conditions $\forall i \in I$: $|B \cap T_i| = 1$ and $\forall i \in I$: $|B \cap U_i| = 1$, i.e. which is a common frame.

Our interest is naturally in finding either minimal frames, or, in the case when the closure space does not have minimal frames, in finding the ways to minimize them according to particular needs. We will study the relationship between different frames from this perspective.

Proposition 2.5. *Let subsets B_1 and B_2 of S be frames in <S,f>, and g_2 be the restriction of closure f to B_2. If $B_1 \subseteq B_2$, then B_1 is a frame in <B_2,g_2>.*

Proof: Let A be a subset of B. Then, $g_2(B_1 \cap g_2(A)) = f(B_1 \cap (f(A) \cap B_2)) \cap B_2 = f(B_1 \cap f(A)) \cap B_2 = f(A) \cap B_2 = g(A)$.

Proposition 2.6. *Let subset B_2 of S be a frame in <S,f> and subset B_1 of B_2 be a frame in <B_2, g_2>, where g_2 is the restriction of closure f to B_2. Then B_1 is a frame in <S,f>.*

Proof: Let A be a subset of S. Then $f(B_1 \cap (f(A)) = f(f(B_1 \cap (f(A))) = f(f(B_1 \cap (B_2 \cap (f(A))))$ $= f(f(B_1 \cap (B_2 \cap f(B_2 \cap f(A))))) = f(f(B_1 \cap g_2(B_2 \cap f(A)))) \supseteq f(g_2(B_1 \cap g_2(B_2 \cap f(A)))) =$ $f(g_2(B_2 \cap f(A))) = f(B_2 \cap f(B_2 \cap f(A))) = f(B_2 \cap f(A)) = f(A)$.

Corollary. *A frame B in a closure space <S,f> is minimal, iff the closure space <B,g> with the restriction g of closure f to B is a simple space.*

3. Closure Space Induced on the Power Set

Examples of dual information systems with different varieties (such as hotel room keys considered as members of the variety of all keys and at the same time as geometric structures integrating the variety of molecules into particular shape) suggest that the dualism of information manifestations is a result of the hierarchic relationship in which members of one variety are varieties themselves. Thus, we can try to inquire the relationship between closure spaces on a set S and on its power set 2^S.

The simplest fact related to the transition from a set to its power set, well known from the very beginning of the study of closure spaces, is that the family of all Moore families (each being a subset of 2^S) is itself a Moore family (as a subset of 2^T, where $T = 2^S$). This fact has interesting consequences for the structure of closure spaces on S. Closure operators form a complete lattice, in which we can consider the least or the greatest closure operator of some property, which is greater than, or less then given one, i.e. we can consider minimal or maximal modifications of closure operators [8]. However, this relationship between a set S and its power set does not seem to build any specific relationship between closure operators on set S and on its power set.

More promising is approach in which the binary relation R is built between the set S and its power set 2^S by the membership of elements of S in the closures of subsets of S:

Definition 2.1. *Let $<S,f>$ be a transitive closure space. Define for $x \in S$ and $A \in 2^S$ a binary relation $R_f \subseteq S\times 2^S$ such that xR_fA if $x \in f(A)$. If no confusion is likely, we will write simply R instead of R_f.*

The Galois connection defined by this relation was used 70 years ago by Everett to demonstrate that every closure space is produced by a Galois connection, as for every subset A of S we have $R_f{}^{*a}R_f{}^{a}(A) = f(A)$, i.e. we recover the original closure operator f [9]. For our purpose, more interesting is the other closure operator defined on the power set of S.

We have: $\forall A\subseteq S$: $R^a(A) = \{B\in 2^S: A \subseteq f(B)\}$ or equivalently $R^a(A) = \{B\in 2^S: f(A) \subseteq f(B)\}$. Also we have: $\forall \beta\subseteq 2^S$: $R^{*a}(\beta) = \cap\{f(B): B\in\beta\}$.

Thus, we can define a transitive closure g on 2^S by:

$\forall \beta\subseteq 2^S$: $g(\beta) = R^aR^{*a}(\beta) = \{A\subseteq S: \cap\{f(B): B\in\beta\} \subseteq f(A)\}$.

We know from the properties of Galois connections that the complete lattice of f-closed subsets of S (f-Cl) is dually isomorphic to the lattice of g-closed subsets of the power set 2^S (g-Cl). This dual isomorphism is given by either φ: $2^S \rightarrow 2^T$, where $T = 2^S$ with φ: $A \rightarrow R^a(A) = \{B\in 2^S: A \subseteq f(B)\}$, or by ψ: $2^T \rightarrow 2^S$, where again $T = 2^S$ with ψ: $\beta \rightarrow R^{*a}(\beta) = \cap\{f(B): B\in\beta\}$.

Now we can observe that g-Cl consists of the families of subsets of S which satisfy the condition: $\forall \beta\subseteq 2^S$: $\beta\in$ g-Cl *iff* $[\forall A\subseteq S: \cap\{f(B): B\in\beta\} \subseteq f(A) \Rightarrow A\in\beta\}$.

Thus, we get a mutual correspondence linking all closure operators on S with some closure operators on the power set of S in such a way that corresponding closure operators have dually isomorphic lattices of closed subsets. This correspondence is our candidate for the formalization of the duality of information manifestations in hierarchically related information systems.

Our present goal is to investigate the properties of the closure operator g and its relation to the closure operator f. We start with an example of the simple example of the trivial closure operator f such that $\forall A\subseteq S$: $f(A) = A$ whose set of closed subsets is the power set of S and which generates the relation xRA iff $x\in A$.

Then we have $\forall A \subseteq S$: $R^a(A) = \{B \in 2^S: A \subseteq B\}$, i.e. $R^a(A)$ is the principal filter generated by A. Also we have: $\forall \beta \subseteq 2^S$: $R^{*a}(\beta) = \cap\{B: B \in \beta\}$. Finally $\forall \beta \subseteq 2^S$: $g(\beta) = \{A \subseteq S: \cap\{B: B \in \beta\} \subseteq A\}$, i.e. g ($\beta$) is the principal filter generated by the intersection of all sets in β. This means that $\forall \beta \subseteq 2^S$: $\beta \in$ g-Cl *iff* $[\forall A \subseteq S: \cap\{B: B \in \beta\} \subseteq A \Rightarrow A \in \beta\}$.

This special and of course trivial case gives the most direct association with the dualism of information manifestations, if information is understood as a property of objects, and the property is identified with the set of objects which have it. But this also demonstrates that the intuitive understanding of the dualism can be misleading, as in more general case the association can be much more complicated.

In the general case where there are no assumptions regarding the closure operator f on S, we have the following simple facts provided here without proofs:

Proposition 3.1.

1. $\varnothing \in \beta \Rightarrow g(\beta) = S$.
2. $B_1, B_2 \in \beta \,\&\, B_1 \leq B_2 \Rightarrow g(\beta) = g(\beta \setminus \{B_2\})$.
3. $B_1, B_2 \in \beta \,\&\, f(B_1) \leq f(B_2) \Rightarrow g(\beta) = g(\beta \setminus \{B_2\})$.
4. $\alpha \subseteq \beta \Rightarrow g(\beta) = g(\beta \setminus \alpha \cup (\cap \alpha))$
5. $(\cap \beta) \in \beta \Rightarrow g(\beta) = g(\{(\cap \beta)\})$.
6. $\beta' = \{f(B): B \in \beta\}) \Rightarrow g(\beta') = g(\beta)$.
7. $A \in g(\beta) \,\&\, A \subseteq B \Rightarrow B \in g(\beta)$ *(i.e. $g(\beta)$ is dually hereditary)*
8. *$g(\beta)$ does not have to be a filter.*
9. $f(A) = A \in g(\beta) \,\&\, f(B) = B \in g(\beta) \Rightarrow A \cap B = f(A) \cap f(B) \in g(\beta)$.
10. *$g(\beta) = \beta \Rightarrow f$-closed elements of β form a filter.*

The last statement is of special interest for us, as we can see that each g-closed family of subsets of S corresponds to some filter of f-closed sets.

The sixth statement of the proposition has even more important consequences.

Corollary

The family f-Cl of f-closed subsets of S is a frame for the closure space $<2^S, g>$, i.e. $\forall \beta \subseteq 2^S\ \exists \beta_A \subseteq f\text{-}Cl \subseteq 2^S: g(\beta) = g(\beta_A)$.

4. Conclusion

The dualism of information manifestations understood as the relationship between information systems arranged into the two level hierarchy, in which the multiplicity of higher rank information carrier consists of elements which themselves are multiplicities of lower rank information systems can be formalized using the Galois connection defined by a binary relation $R_f \subseteq S \times 2^S$ such that $xR_f A$ if $x \in f(A)$. The closure relation f on the set S describes the information system of the lower rank.

The Galois closure $A \rightarrow R_f^{*a}R_f^a(A)$ for the subsets of S turns out to be simply closure f. The other Galois closure on the power set of S defined by $\forall \beta \subseteq 2^S$: $g(\beta) = R^aR^{*a}(\beta) = \{A \subseteq S: \cap\{f(B): B \in \beta\} \subseteq f(A)\}$ describes the information system of higher rank. The relationship between closures makes the family of f-closed subsets a frame for g closure. The g-closed subfamilies of the power set correspond to filters in the family of f-closed subsets. The relationship clarifies the meaning of the families of closed subsets and of the filters in these families introduced ad hoc by the author in his earlier papers on the formalism for information theory.

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